QUASIOPTICAL APPROXIMATION FOR A BUNDLE OF SHEAR WAVES IN A NONLINEAR HEREDITARY MEDIUM

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Diffraction, dispersion, and dissipative effects appear in an intensive wave bundle being propagated in a medium with heredity in addition to the nonlinear effects. In the most interesting zone, where all the listed effects have a mutually commensurate influence, a quasi-optical approximation can be utilized to describe the wave process. On the basis of this approximation, the authors of [1] derived an equation for acoustic waves in a fluid. A similar approach was also used in considering longitudinal waves in solids and plates [2, 3].

In this paper an approximate integrodifferential equation is derived that describes quasiplanar shear wave propagation in a solid with heredity.

Nonlinear strain waves in a medium with linear heredity are described by the equations

$$\rho \ddot{u}_i = L_{ij,j}; \tag{1}$$

$$L_{ij} = \frac{\partial}{\partial u_{i,j}}\Big[\frac{\lambda}{2}(\varepsilon_{kk})^2 + \mu\varepsilon_{lk}\varepsilon_{kl} + \frac{\nu_1}{6}(\varepsilon_{kk})^3 + \nu_2\varepsilon_{kk}\varepsilon_{ln}\varepsilon_{nl} + \frac{4}{3}\nu_3\varepsilon_{ln}\varepsilon_{nk}\varepsilon_{kl} + \gamma_1(\varepsilon_{kk})^4 + \gamma_2(\varepsilon_{kk})^2\varepsilon_{ln}\varepsilon_{nl} + \gamma_3(\varepsilon_{lk}\varepsilon_{kl})^2$$
$$+ \gamma_4\varepsilon_{kk}\varepsilon_{lm}\varepsilon_{mn}\varepsilon_{nl}\Big] - \delta_{ij}\int_{-\infty}^{t} K_1(t-\xi)\,u_{k,k}(\xi)\,d\xi - \int_{-\infty}^{t} K_2(t-\xi)\,u_{i,j}(\xi)\,d\xi, \quad i,\ j,\ k,\ l,\ m,\ n = 1,\ 2,\ 3, \tag{2}$$

where L_{ij} are the Lagrange stress tensors, $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j})$ is the finite strain tensor, u_i is the displacement vector component, $K_1(t)$ and $K_2(t)$ are the bulk and shear relaxation kernels, respectively. We shall consider a plane strain field when the displacement vector lies in the xOy plane. Let a bundle of shear waves be propagated along the x axis. For convenience in the subsequent exposition, we will write down the linear part of the system (1) and and (2) in detail:

$$\rho\ddot{u}_1 - (\lambda+\mu)(u_{1,xx} + u_{2,xy}) - \mu(u_{1,xx} + u_{1,yy}) + \int_{-\infty}^{t}[K_1(t-\xi) + K_2(t-\xi)]$$
$$\times[u_{1,xx}(\xi) + u_{2,xy}(\xi)]\,d\xi + \int_{-\infty}^{t} K_2(t-\xi)[u_{1,xx}(\xi) + u_{1,yy}(\xi)]\,d\xi = f_1; \tag{3}$$

$$\rho\ddot{u}_2 - (\lambda+\mu)(u_{2,yy} - u_{1,xy}) - \mu(u_{2,xx} + u_{2,yy}) + \int_{-\infty}^{t}[K_1(t-\xi) + K_2(t-\xi)]$$
$$\times[u_{2,yy}(\xi) + u_{1,xy}(\xi)]\,d\xi + \int_{-\infty}^{t} K_2(t-\xi)[u_{2,xx}(\xi) + u_{2,yy}(\xi)]\,d\xi = f_2. \tag{4}$$

Here f_1 and $f_2 = (3/2)(\lambda + 2\mu + 2\nu_2 + 4\nu_3 + 2\gamma_3)(u_{2,x})^2u_{2,xx} + f_{22}$ are the nonlinear components, where only the component governing the nonlinear effects along bundle propagation is written down explicitly. To derive the approximate equation of a wave bundle being propagated along the x axis, we utilize a representation of diffraction as the transverse diffusion of amplitude. We introduce the ray variables associated with the wave being propagated [1, 3-5]:

$$\tau = t - x/c,\ \eta = \varepsilon^{1/2}y,\ \chi = \varepsilon x$$

(c is the characteristic wave velocity, not known in advance), and we expand the displacements in a power series in the small nonlinearity parameter ε

$$u_1 = \varepsilon^{1/2}(u_1^1 + \varepsilon u_1^2 + \ldots),\quad u_2 = u_2^0 + \varepsilon u_2^1 + \ldots, \tag{5}$$

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where $\varepsilon = 3|\lambda + 2\mu + 2\nu_2 + 4\nu_3 + 2\gamma_3|A^2/2\mu l^2$; A is the maximal amplitude of the displacement u_2 and l is the wavelength. The selection of the variables is explained by the fact that all the quantities, both along and across the direction of bundle propagation, vary because of bundle divergence, nonlinearity, and heredity. Changes across the bundle occur more rapidly. Substituting the asymptotic expansions (5) into (3) and (4), we retain just terms in linear powers of ε

$$\rho u^0_{2,\tau\tau} - \frac{\mu}{c^2} u^0_{2,\tau\tau} = 0; \tag{6}$$

$$\rho u^1_{1,\tau\tau} - \frac{(\lambda+\mu)}{c^2}\left(u^1_{1,\tau\tau} - c u^0_{2,\tau\eta}\right) - \frac{\mu}{c^2} u^1_{1,\tau\tau} = 0; \tag{7}$$

$$\rho u^1_{2,\tau\tau} - (\lambda+\mu)\left(u^0_{2,\eta\eta} - \frac{1}{c} u^1_{1,\tau\eta}\right) - \mu\left(\frac{1}{c^2} u^1_{2,\tau\tau} - \frac{2}{c} u^0_{2,\tau\chi} + u^0_{2,\eta\eta}\right) + $$
$$+ \frac{1}{c^2}\int\limits_{-\infty}^{\tau} K_2(\tau-\xi)\, u^0_{2,\tau\tau}(\xi)\, d\xi = -\frac{3}{2}\frac{(\lambda+2\mu+2\nu_2+4\nu_3+2\gamma_2)}{c^3\varepsilon}\left(u^0_{2,\tau}\right)^2 u^0_{2,\tau\tau}. \tag{8}$$

An expression for the velocity $c = \sqrt{\mu/\rho}$ follows from the relationship (6) and the relationship between the strains $u^1_{1,\tau}$ and $u^0_{2,\eta}$:

$$u^1_{1,\tau} - c u^0_{2,\eta} = 0, \tag{9}$$

from (7), which reflects the fact that the wave being propagated remains a pure shear wave in the zeroth and first approximation, i.e., $\operatorname{div} u = 0$. Substituting (9) into (8), we obtain a single scalar equation for the bundle

$$u^0_{2,\tau\chi} + \alpha\left(u^0_{2,\tau}\right)^2 u^0_{2,\tau\tau} + \int\limits_{-\infty}^{\tau} G(\tau-\xi)\, u^0_{2,\tau\tau}(\xi)\, d\xi = \gamma u^0_{2,\eta\eta} \tag{10}$$

or using the notation $u_{2,\tau} = \psi$, we have

$$\frac{\partial}{\partial\tau}\left(\psi_{,\chi} + \alpha\psi^2\psi_{,\tau} + \int\limits_{-\infty}^{\tau} G(\tau-\xi)\,\psi_{,\tau\tau}(\xi)\, d\xi\right) = \gamma\psi_{,\eta\eta}, \tag{11}$$

where

$$\alpha = \frac{3(\lambda+2\mu+2\nu_2+4\nu_3+2\gamma_3)}{4\rho c^3\varepsilon};\quad G(t) = K_2(t)/\rho;$$

$$\gamma = \left(2c_l^2 - c^2\right)/2;\quad c_l^2 = (\lambda+2\mu)/\rho.$$

As is seen from the expressions presented, $\gamma > 0$ always, while $\alpha > 0$ in the majority of cases. Let $\alpha > 0$; then by replacing the dependent variable $\varphi = \psi\sqrt{\alpha}$ we reduce Eq. (11) to the form

$$\frac{\partial}{\partial\tau}\left(\varphi_{,\chi} + \varphi^2\varphi_{,\tau} + \int\limits_0^{\infty} G(\xi)\,\varphi_{,\tau}(\tau-\xi)\, d\xi\right) = \gamma\varphi_{,\eta\eta}. \tag{12}$$

For the classical model of a Voigt viscoelastic medium $(G(t) - \varkappa_1\dot{\delta}(t))$ the integrodifferential equation (12) becomes a differential equation

$$\frac{\partial}{\partial\tau}\left(\varphi_{,\chi} + \varphi^2\varphi_{,\tau} + \varkappa_1\varphi_{,\tau\tau}\right) = \gamma\varphi_{,\eta\eta}. \tag{13}$$

In the case of a Maxwell viscoelastic medium [a viscoelastic fluid $G(t) = \varkappa_2 e^{-\beta t}$], the integral operator can also be replaced by a differential operator when the relaxation time $1/\beta$ is much less than the characteristic time of wave variation. To do this we expand the desired function $\varphi(\tau - z)$ in a Taylor series in the neighborhood of

$$\varphi_{,\tau}(\tau - z) = \varphi_{,\tau}(\tau) - \varphi_{,\tau\tau}(\tau) z + \varphi_{,\tau\tau\tau}(\tau) z^2/2 + \ldots$$

and we represent the integral term in (12) in the form

$$\varkappa_2 \int\limits_{-\infty}^{\tau} \exp[\beta(\xi-\tau)]\,\varphi_{,\tau}(\xi)\, d\xi = \frac{\varkappa_2}{\beta}\varphi_{,\tau} - \frac{\varkappa_2}{\beta^2}\varphi_{,\tau\tau} + \ldots$$

Limiting ourselves to two terms in the asymptotic expansion, we write the equation for a shear wave bundle in a hereditary medium with an exponential kernel:

$$\frac{\partial}{\partial \tau}\left[\varphi_{,\chi}+\left(-\frac{\varkappa_2}{\beta}+\varphi^2\right)\varphi_{,\tau}-\frac{\varkappa_2}{\beta^2}\varphi_{,\tau\tau}\right]=\gamma\varphi_{,\eta\eta}. \quad (14)$$

Within the framework of (12)-(14), we can examine a broad circle of phenomena that are due to bundle self-action and extrinsic plane waves: nonlinear refraction, self-focusing, nonlinear waveguide propagation, etc. These equations contain a cubic nonlinearity that distinguishes them from corresponding equations of sound bundles in a fluid and gas [4, 6], where a quadratic nonlinearity plays a governing role. Therefore, exactly as in optics, self-focusing effects can be expected in solids for bundles of quasiharmonic shear waves. In this case it is convenient to work with the equation for the slowly varying complex amplitude $A(\chi, \eta)$

$$u_2^0 = A(\chi, \eta)\exp(-i\omega\tau) + \text{c.c.} \quad (15)$$

Substituting (15) into (10), we find that the amplitude should satisfy the nonlinear Schroedinger equation

$$2i\omega\frac{\partial A}{\partial \chi}+\alpha\omega^4|A|^2A+\gamma\frac{\partial^2 A}{\partial \eta^2}=-\omega^2\tilde{G}A, \quad (16)$$

where $\tilde{G}(\omega)$ is the Fourier transform of the kernel $G(\xi)$ in a semi-infinite interval

$$\tilde{G}(\omega)=\int_0^\infty G(\xi)\exp(i\omega\xi)\,d\xi.$$

In the case of a Voigt medium, the pure dissipation term $-i\omega^3 A/2$ will be in the first part of Eq. (16).

The cubic Schroedinger equation with zero right side has been investigated in sufficient detail in connection with the self-focusing problem for two-dimensional bundles and the formation of envelope waves in nonlinear media [7]. The right side in (16) can be considered a small perturbation for whose analysis distinct approximate methods have been developed that are based particularly on the method of the inverse scattering problem (see [8, 9], say).

In conclusion, we note that singular heredity kernels are also utilized to describe wave processes in solid media, where it is proposed to select the singular kernels with a singularity not stronger than a logarithmic singularity [10].

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